CRITICAL LOADS OF NONIDEAL SHALLOW CYLINDRICAL SHELLS

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The presence of small initial geometric imperfections of shell structures causes premature buckling. Great attention to investigations on the sensitivity of the critical load to the initial geometric imperfections has been paid in [1]. It is mentioned in papers of Seide, Arbosh, Kaplan, and Babcock, Jr. [2] that precisely the initial deflections are the main cause for the huge spread in experimental results and the poor correlation between theoretical and experimental results. The importance of the question of the degree of sensitivity to initial imperfections is also noted by Fersht [2].

By perturbation theory methods, problems of the linear theory of the buckling of nonideal shallow cylindrical shells are investigated in this paper. Shells are examined that are loaded by transverse and hydrostatic pressure as also longitudinally compressed shells. The eigennumbers (critical loads) and eigenfunctions (buckling modes) of nonideal shells are sought in the form of asymptotic series in a small parameter that characterizes the amplitude of the initial imperfections. Explicit formulas are obtained for the first terms of the expansions. Only the initial linear system of equations of shallow shell theory is used in the method proposed for the construction of the mentioned eigennumbers and eigenfunctions.

1. FORMULATION OF THE PROBLEM. CONSTRUCTION OF ASYMPTOTIC EXPANSIONS OF EIGENFUNCTIONS AND NUMBERS

Distortion of the spectrum in linear stability problems is studied for nonideal cylindrical shells where the spectrum in stability problems of ideal cylindrical shells is taken as basis; the system of equations for nonideal shells in dimensionless notation has the form [3]

$$\varepsilon^2\Delta\Delta w + f_{xx} - \lambda\left(a_1 w_{xx} + a_2 w_{yy}\right) - \mu\,(h/R)\left(f_{yy}w^0_{xx} + f_{xx}w^0_{yy} - 2f_{xy}w^0_{xy}\right) = 0, \quad (1.1)$$
$$\Delta\Delta f = w_{xx} - \mu\,(h/R)\left(w_{xx}w^0_{yy} + w_{yy}w^0_{xx} - 2w_{xy}w^0_{xy}\right),$$

$$\Delta w = w_{xx} + w_{yy},\quad f^0_{xx} = \lambda\,(R/h)\,a_2,\quad f^0_{yy} = \lambda\,(R/h)\,a_1,\quad f^0_{xy} \equiv 0,$$
$$w = \frac{w_*}{h},\quad w^0 = \frac{w^0_*}{h},\quad f = \frac{f_*}{Eh^2R},\quad \varepsilon^2 = \frac{1}{12\,(1-\nu^2)}\left(\frac{h}{R}\right)^2,\quad \begin{matrix}0 \leqslant x \leqslant L/R,\\ 0 \leqslant y \leqslant 2\pi,\end{matrix}$$

where w and w_* are the dimensionless and dimensional normal deflections, f and f_* are dimensionless and dimensional stress functions, w^0 and w^0_* are dimensionless and dimensional functions characterizing the initial imperfections of the shell, where $\max|w^0| = 1$, μ is a small parameters proportional to the amplitude of the initial imperfections, h, R, L are the thickness, radius, and length of the cylindrical shell, ε is a small parameter governing the thin-walledness of the structure, E and ν are Young's modulus and the Poisson ratio, the constant force components in the longitudinal f^0_{xx} and circumferential f^0_{yy} directions are proportional to the loading parameter λ, a_1, a_2 are coefficients. Shells are considered that are loaded by a constant transverse pressure (problem 1, $a_2 = -1$, $a_1 = 0$), by a constant hydrostatic pressure (problem 2, $a_2 = -1$, $a_1 = -1/2$), and longitudinally compressed shells (problem 3, $a_2 = 0$, $a_1 = -1$).

We assume: 1) such loading and clamping conditions are realized on the shell endfaces that for $\mu \equiv 0$ we have a membrane state of stress; 2) out of the four boundary conditions for each of the endfaces two of the linear conditions are formulated only for the normal deflection w and the other two linear conditions for only the stress function f, i.e.,

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$$g_{pj}f = 0 \ (p = 0, 1, j = 1, 2), \ g_{pj}w = 0 \ (p = 0, 1, j = 3, 4). \tag{1.2}$$

Here g_{pj} are differential operators whose form remains invariant for $\mu \equiv 0$ and $\mu \neq 0$. The second constraint is used substantially for the construction of perturbation theory methods [4-6] for eigenfunctions and eigenvalues (μ is a small parameter), and namely those values of the loading parameter λ_i^* are sought for which the system of equations (1.1) with the appropriate boundary conditions (1.2) has the nontrivial solution w_i^* and f_i^*:

$$\begin{gathered}\varepsilon^2 aw_i^* + df_i^* - \mu sf_i^* - \lambda_i^* bw_i^* = 0, \quad af_i^* = dw_i^* - \mu tw_i^*, \\ g_{pj}f_i^* = 0 \ (p = 0, 1, \ j = 1, 2), \quad g_{pj}w_i^* = 0 \ (p = 0, 1, \ j = 3, 4), \\ a = \Delta\Delta, \quad d = (\ )_{xx}, \quad s = (h/R)\left[w_{xx}^0(\ )_{yy} + w_{yy}^0(\ )_{xx} - 2w_{xy}^0(\ )_{xy}\right], \\ b = a_1(\ )_{xx} + a_2(\ )_{yy}, \quad t = (h/R)\left[w_{yy}^0(\ )_{xx} + w_{xx}^0(\ )_{yy} - 2w_{xy}^0(\ )_{xy}\right].\end{gathered} \tag{1.3}$$

The eigenfunction and eigennumber problem (1.3) for $\mu \neq 0$ is called perturbed [4-6]; for $\mu \equiv 0$ this problem goes over into the unperturbed problem

$$(\varepsilon^2 a^2 + d^2)\Phi_i - \lambda_i ba\Phi_i = 0, \ G_{pj}\Phi_i = 0 \ (p = 0, 1, j = 1, 2, 3, 4), \tag{1.4}$$

where G_{pj} are operators of the boundary conditions (1.2), λ_i and Φ_i are the eigennumbers and eigenfunctions of the unperturbed problem in terms of which the normal deflections and stress functions corresponding to the unperturbed problem (1.4) are expressed by means of $w_i = a\Phi_i$, $f_i = d\Phi_i$. Following the customary procedures [4-6], we represent the eigenfunctions w_i^*, f_i^* and the numbers λ_i^* of the problem (1.3) in the form of asymptotic series in the parameter μ:

$$\begin{gathered}w_i^* = w_i^{(0)} + \mu w_i^{(1)} + \mu^2 w_i^{(2)} + \ldots, \quad (bw_i^*, \Phi_j^*) = \delta_{ij}, \quad a\Phi_i^* = w_i^*, \\ f_i^* = f_i^{(0)} + \mu f_i^{(1)} + \mu^2 f_i^{(2)} + \ldots, \quad (ba\Phi_i, \Phi_j) = \delta_{ij}, \quad ((\varepsilon^2 a^2 + d^2)\Phi_i, \Phi_j) = \lambda_i \delta_{ij}, \\ \lambda_i^* = \lambda_i + \mu\lambda_i^{(1)} + \mu^2\lambda_i^{(2)} + \ldots, \quad i = 1, 2, \ldots, \quad 0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots\end{gathered} \tag{1.5}$$

The normalization conditions for the perturbed problem (1.3) are presented in the first line of (1.5), the parentheses denote the scalar product, normalization conditions for the unperturbed problem are in the second line (δ_{ij} are Kronecker symbols), and the eigenvalues λ_i of the unperturbed problem (1.4) are ordered in the usual manner. The asymptotic expansions (1.5) are substituted into the system of equations and boundary conditions of the problem (1.3):

$$\begin{gathered}\varepsilon^2 a\left(w_i^{(0)} + \mu w_i^{(1)} + \ldots\right) + (d - \mu s)\left(f_i^{(0)} + \mu f_i^{(1)} + \ldots\right) - \left(\lambda_i + \mu\lambda_i^{(1)} + \ldots\right)\times \\ \times b\left(w_i^{(0)} + \mu w_i^{(1)} + \ldots\right) = 0, \quad g_{pj}\left(w_i^{(0)} + \mu w_i^{(1)} + \ldots\right) = 0 \quad (p = 0, 1, \ j = 3, 4);\end{gathered} \tag{1.6}$$

$$a\left(f_i^{(0)} + \mu f_i^{(1)} + \ldots\right) = (d - \mu t)\left(w_i^{(0)} + \mu w_i^{(1)} + \ldots\right), \quad g_{pj}\left(f_i^{(0)} + \mu f_i^{(1)} + \ldots\right) = 0 \quad (p = 0, 1, \ j = 1, 2). \tag{1.7}$$

We collect term of the same order of smallness in problems (1.6) and (1.7). We then start with terms with μ^0, the problem of these terms agrees with problem (1.4) after appropriate manipulation, the eigennumbers and eigenfunctions are λ_i and Φ_i ($i = 1, 2, \ldots$), where the latter are orthonormalized. Equating terms in μ to the first power, we have

$$af_i^{(1)} = dw_i^{(1)} - tw_i^{(0)}, \quad g_{pj}f_i^{(1)} = 0 \quad (p = 0, 1, \ j = 1, 2); \tag{1.8}$$

$$\begin{gathered}\varepsilon^2 aw_i^{(1)} + df_i^{(1)} - sf_i^{(0)} - \lambda_i bw_i^{(1)} - \lambda_i^{(1)} bw_i^{(0)} = 0, \\ g_{pj}w_i^{(1)} = 0 \quad (p = 0, 1, \ j = 3, 4).\end{gathered} \tag{1.9}$$

The other problems are formulated in the same manner. The main difficulty in solving the problems (1.8) and (1.9) is related to the fact that the unknown functions $f_i^{(1)}$ and $w_i^{(1)}$ are in the equations (1.8) and (1.9).

Construction of the expansion (1.5) depends on the multiplicity of the eigennumber λ_i of the problem (1.4): 1) λ_i is prime; 2) λ_i has the multiplicity $i_{00} - i_0 + 1$, i.e.,

$$\lambda_{i_0} = \lambda_{i_0+1} = \ldots = \lambda_{i_{00}}. \tag{1.10}$$

In general, the eigenvalues for closed cylindrical shells are always duplex; however, we shall not eliminate the first case from consideration since it is typical for unclosed cylindrical shells (panels) whose rectilinear edges are hinge-supported.

2. PRIME EIGENVALUE

Thus, λ_i is prime, then $w_i^{(0)} = w_i$, $f_i^{(0)} = f_i$, we recall that w_i and f_i are expressed in terms of the eigenfunctions Φ_i. Let us assume [4-6] that the functions $f_i^{(1)}$ and $w_i^{(1)}$ are expanded in a series of the eigenfunctions f_i and w_i, more exactly

$$f_i^{(1)} = \sum_{k=1}^{\infty} \alpha_{ki} f_k, \quad w_i^{(1)} = \sum_{k=1}^{\infty} \beta_{ki} w_k. \tag{2.1}$$

It follows from the normalization condition (1.5) that $\beta_{ii} = 0$. The eigenfunctions f_i and w_i satisfy the boundary-value conditions of problems (1.8) and (1.9); hence, there remains just to select the coefficients α_{ki} and β_{ki}. We express the functions f_i and w_i in terms of Φ_i; then from the equations of the problem (1.8) we obtain relationships connecting the coefficients α_{ki} and β_{ki}:

$$\sum_{k=1}^{\infty} (\alpha_{ki} - \beta_{ki}) (ad\Phi_k, \Phi_j) = -(ta\Phi_i, \Phi_j), \quad j = 1, 2, \ldots \tag{2.2}$$

Let us go over to an investigation of the infinite system (2.2) for the first and second buckling problems; let us recall the structure of the solution governing the buckling mode: The solution consists of the fundamental (generating) part and the edge effect for stability [7, 8]. Hence, the elements outside the main diagonal of the determinant corresponding to the system (2.2) characterize the rapidity of the damping of the edge effects for stability, i.e.,

$$\sum_{k \neq j} (\alpha_{ki} - \beta_{ki}) (ad\Phi_k, \Phi_j) \approx 0. \tag{2.3}$$

Let us omit these secondary elements outside the diagonal of the determinant; the coefficients α_{ki} are expressed in terms of β_{ki}:

$$\alpha_{ki} = \beta_{ki} - (ta\Phi_i, \Phi_k)(ad\Phi_k, \Phi_k)^{-1}, \; k = 1, 2, \ldots \tag{2.4}$$

The relationships (2.1) are taken into account in determining the coefficients β_{ki} and $\lambda_i^{(1)}$, as are the formulas (2.4) and a relationship analogous to (2.3)

$$\sum_{k \neq j} (\alpha_{ki} - \beta_{ki}) \left(d^2\Phi_k, \Phi_j\right) \approx 0.$$

We finally obtain

$$\lambda_i^{(1)} = \alpha_{ii} \left(d^2\Phi_i, \Phi_i\right) - (sd\Phi_i, \Phi_i),$$
$$\beta_{ki} = \frac{(sd\Phi_i, \Phi_k) + (ta\Phi_i, \Phi_k)(ad\Phi_k, \Phi_k)^{-1} \left(d^2\Phi_k, \Phi_k\right)}{\lambda_k - \lambda_i} \quad \text{for } i \neq k. \tag{2.5}$$

In the general case $\alpha_{ii} \neq 0$ although $\beta_{ii} = 0$. The approximate relationships (2.4) and (2.5) are transformed into exact relations for all three problems under consideration if the shells are hinge-supported along the contour. Higher terms of the expansion can also be constructed in an analogous manner.

3. MULTIPLE EIGENVALUE

Now, let λ_i be a multiple eigenvalue (see (1.10)). The eigenfunctions $w_i^{(0)}$ and $f_i^{(0)}$ are linear combinations of the unperturbed eigenfunctions [4-6] corresponding to a multiple eigenvalue (only hinge-supported shells are considered henceforth):

$$w_i^{(0)} = a\chi_i, \quad f_i^{(0)} = d\chi_i, \quad \chi_i = \sum_{n=i_0}^{i_{00}} \rho_{ni}\Phi_n, \quad i = i_0, i_0+1, \ldots, i_{00}. \tag{3.1}$$

The new eigenfunctions $w_i^{(0)}$, $f_i^{(0)}$ differ from the eigenfunctions of the unperturbed problem in that the former are adapted to the perturbation mode by the selection of the system of coefficients ρ_{ni} (n, $i = i_0, i_0 + 1, \ldots, i_{00}$). A linear homogeneous system is obtained for the unknown coefficients ρ_{ni}:

$$\left(\sum_{n=i_0}^{i_{00}} \rho_{ni} ta\Phi_n, \Phi_k\right)(ad\Phi_k, \Phi_k)^{-1}\left(d^2\Phi_k, \Phi_k\right) + \left(\sum_{n=i_0}^{i_{00}} \rho_{ni} sd\Phi_n, \Phi_k\right) + \tag{3.2}$$
$$+ \lambda_i^{(1)}\rho_{ki} = 0, \quad k = i_0, i_0+1, \ldots, i_{00}.$$

We determine $\lambda_i^{(1)}$ from the solvability condition for the system (3.2). We assume A) that all $\lambda_i^{(1)}$, $i = i_0, i_0 + 1, \ldots, i_{00}$ are distinct, and B) $\lambda_{j_0}^{(1)} = \lambda_{j_0+1}^{(1)} = \ldots = \lambda_{j_{00}}^{(1)}, i_0 \leqslant j_0 < j_{00} \leqslant i_{00}$.

Case A. Let the multiple eigenvalue be split into simple eigenvalues in a first approximation. For each $\lambda_i^{(1)}$ we obtain a system of coefficients ρ_{ni} (n, $i = i_0, i_0 + 1, \ldots, i_{00}$) from the homogeneous system (3.2) and the normalization conditions (see the first line in the relations (1.5)). Thus, the zeroth approximation (3.1) is constructed for the eigenfunctions.

Let us go over to terms of first order smallness $f_i^{(1)}$ and $w_i^{(1)}$; these functions are expanded into the series (2.1), certain terms of which are zero $\beta_{ki} \equiv 0$, k, $i = i_0, i_0 + 1, \ldots, i_{00}$ (see (1.5)). The formulas analogous to (2.4) and (2.5) finally are

$$\alpha_{ki} = \beta_{ki} - (ta\chi_i, \Phi_k)(ad\Phi_k, \Phi_k)^{-1}, \quad k = 1, 2, \ldots,$$
$$\beta_{ki} = \frac{(sd\chi_i, \Phi_k) + (ta\chi_i, \Phi_k)(ad\Phi_k, \Phi_k)^{-1}\left(d^2\Phi_k, \Phi_k\right)}{\lambda_k - \lambda_i}, \quad k \neq i_0, i_0+1, \ldots, i_{00}.$$

Terms of higher order of smallness in the expansions (1.5) are constructed in the same manner also. Let us note that the method proposed for constructing these asymptotic expansions differs from the method in [8] since no information is used here except the initial system of equations (1.1).

Case B. Let $\lambda_{j_0}^{(1)} = \lambda_{j_0+1}^{(1)} = \ldots = \lambda_{j_{00}}^{(1)}$, $i_0 \leqslant j_0 < j_{00} \leqslant i_{00}$, i.e., part (or all) of the multiple eigenvalues remain multiple. Let us introduce new eigenfunctions and functions of first order of smallness:

$$w_j^{(0)} = \sum_{n=j_0}^{j_{00}} \rho'_{nj} w_n^{(0)}, \quad w_j^{(1)} = \sum_{n=j_0}^{j_{00}} \rho'_{nj} w_n^{(1)},$$
$$f_j^{(1)} = \sum_{n=j_0}^{j_{00}} \rho'_{nj} f_n^{(1)}, \quad j = j_0, j_0+1, \ldots, j_{00},$$

where the functions $w_n^{(0)}$, $w_n^{(1)}$, and $f_n^{(1)}$ are taken from (2.1) and (3.1). It is easy to see that the functions thus constructed satisfy the problems (1.4), (1.8), and (1.9). Analogously to (2.1), we represent the functions

$$w_j^{(2)} = \sum_{k=1}^{\infty} \beta'_{kj} w_k, \quad f_j^{(2)} = \sum_{k=1}^{\infty} \alpha'_{kj} f_k.$$

We obtain a linear homogeneous system analogous to (3.2) for the unknown coefficients ρ'_{nj}:

$$\left(\sum_{n=j_0}^{j_{00}} \rho'_{nj} t w_n^{(1)}, \Phi_k\right) (ad\Phi_k, \Phi_k)^{-1} (d^2\Phi_k, \Phi_k) + \left(\sum_{n=j_0}^{j_{00}} \rho'_{nj} s f_n^{(1)}, \Phi_k\right) +$$

$$+ \lambda_j^{(1)} \sum_{n=j_0}^{j_{00}} \rho'_{nj} \beta_{kn} + \lambda_j^{(2)} \sum_{n=j_0}^{j_{00}} \rho'_{nj} \rho_{kn} = 0, \quad k = j_0, j_0+1, \ldots, j_{00}.$$

The formulas to determine the coefficients α'_{kj} and β'_{kj} have the form

$$\alpha'_{kj} = \beta'_{kj} - \left(\sum_{n=j_0}^{j_{00}} \rho'_{nj} t w_n^{(1)}, \Phi_k\right) (ad\Phi_k, \Phi_k)^{-1}, \quad k = 1, 2, \ldots,$$

$$\beta'_{kj} = \frac{\left(\sum_{n=j_0}^{j_{00}} \rho'_{nj} s f_n^{(1)}, \Phi_k\right) + \left(\sum_{n=j_0}^{j_{00}} \rho'_{nj} t w_n^{(1)}, \Phi_k\right) (ad\Phi_k, \Phi_k)^{-1} (d^2\Phi_k, \Phi_k) + \lambda_j^{(1)} \sum_{n=j_0}^{j_{00}} \rho'_{nj} \beta_{kn}}{\lambda_k - \lambda_j},$$

$$k \neq i_0, \quad i_0 + 1, \ldots, i_{00}.$$

The coefficients $\beta'_{kj} \neq 0$, $k = i_0, i_0 + 1, \ldots, i_{00}$ (see (1.5)).

Investigation of the spectrum in linear stability problems of nonideal shells is the first necessary link in the construction of solutions of nonideal system buckling under loads close to the critical [10, 11].

4. CERTAIN SPECIFIC EXAMPLES FOR THE FIRST AND SECOND PROBLEMS

A. Let the initial imperfection be (x is the longitudinal, and y the circumferential coordinate)

$$w^0 = w^0(x) \quad \left(w_{yy}^0 = 0, \quad w_{xy}^0 = 0\right). \tag{4.1}$$

Taking (4.1) into account, we write the operators s and t for the selected imperfection in the form (see (1.3))

$$s \equiv t \equiv (h/R) w_{xx}^0 \cdot (\;)_{yy}. \tag{4.2}$$

Formulas (2.4) and (2.5) with (4.2) taken into account can be utilized in computations if $\lambda_{nm} \neq \lambda_{n+1,m}$, and moreover the orthogonality of the functions sin ny and cos ny is taken into account. For closed hinge-supported cylindrical shells the eigennumbers λ_{nm} and eigenfunctions are determined in the form (see (1.4))

$$\lambda_{nm} = \left\{\varepsilon^2 \left[n^2 + \left(\frac{m\pi R}{L}\right)^2\right]^4 + \left(\frac{m\pi R}{L}\right)^4\right\} \left[n^2 + \left(\frac{m\pi R}{L}\right)^2\right]^{-2} \left[-a_2 n^2 - a_1 \left(\frac{m\pi R}{L}\right)^2\right]^{-1}, \tag{4.3}$$

$$\Phi_{nm} = \gamma_{nm} \begin{Bmatrix} \sin ny \\ \cos ny \end{Bmatrix} \sin m\pi Rx/L, \quad n = 2, 3, 4, \ldots, m = 1, 2, 3, \ldots,$$

where n is the number of waves in the circumferential direction, m is the number of half-waves in the longitudinal direction, and γ_{nm} are normalizing factors that are determined from the relationships (see the second line in (1.5)):

$$\gamma_{nm}^2\left[\left(\frac{m\pi R}{L}\right)^2+n^2\right]^2\left[-a_1\left(\frac{m\pi R}{L}\right)^2-a_2n^2\right]\int_0^{2\pi}\int_0^{L/R}\sin^2 ny\sin^2\frac{m\pi R}{L}x\,dx\,dy=1.$$

The coefficients α_{kjnm} and β_{kjnm} of the expansion (2.1) are computed from the formulas

$$\alpha_{kjnm}=\beta_{kjnm}-\frac{h}{R}\frac{\left(w_{xx}^0\Delta\Delta\Phi_{nm,yy},\Phi_{kj}\right)}{\left(\Delta\Delta\Phi_{nm,xx},\Phi_{kj}\right)},$$
$$\beta_{kjnm}=\frac{h}{R}\left\{\frac{\left(w_{xx}^0\Delta\Delta\Phi_{nm,yy},\Phi_{kj}\right)\left(\Phi_{kj,xxxx},\Phi_{kj}\right)}{\left(\Delta\Delta\Phi_{kj,xx},\Phi_{kj}\right)}+\right.$$
$$\left.+\left(w_{xx}^0\Phi_{nm,xxyy},\Phi_{kj}\right)\right\}\frac{1}{\lambda_{kj}-\lambda_{nm}}\text{ for } k\neq n \text{ or } j\neq m,$$
$$\beta_{kjnm}=0\text{ for } k=n,\ j=m. \qquad (4.4)$$

Taking (4.4) into account, the direct correction $\lambda_{nm}^{(1)}$ to the accuracy of a factor μ to the eigenvalue λ_{nm} (4.3) has the form

$$\lambda_{nm}^{(1)}=-\frac{h}{R}\left\{\frac{\left(w_{xx}^0\Delta\Delta\Phi_{nm,yy},\Phi_{nm}\right)\left(\Phi_{nm,xxxx},\Phi_{nm}\right)}{\left(\Delta\Delta\Phi_{nm,xx},\Phi_{nm}\right)}+\left(w_{xx}^0\Phi_{nm,xxyy},\Phi_{nm}\right)\right\}.$$

The appearance of the factor h/R in the last formula is related to the units introduced earlier for the measurement of the amplitude of the initial imperfections in (4.2), the shell thickness is selected as the measurement unit. The last formula can be used if the shell is shallow in each half-wave of the initial imperfection (see the initial system of equations (1.1)).

B. Let the initial imperfection be the same as in example A: $w^0 = w^0(x)$, ($w_{yy}^0 = 0$, $w_{xy}^0 = 0$) but $\lambda_{nm} = \lambda_{n+1,m}$. We equate the determinant corresponding to the system (3.2) to zero; after manipulations we have

$$\begin{vmatrix} 2n^2\left(\frac{m\pi R}{L}\right)^2\frac{h}{R}\left(w_{xx}^0\Phi_{nm},\Phi_{nm}\right)+\lambda^{(1)} & 0 \\ 0 & 2(n+1)^2\left(\frac{m\pi R}{L}\right)^2\frac{h}{R}\left(w_{xx}^0\Phi_{n+1,m},\Phi_{n+1,m}\right)+\lambda^{(1)}\end{vmatrix}=0.$$

Since $(w_{xx}^0\Phi_{nm},\Phi_{nm}) = (w_{xx}^0\Phi_{n+1,m},\Phi_{n+1,m})$, then $\lambda_{nm}^{(1)}/\lambda_{n+1,m}^{(1)} = n^2/(n+1)^2 \neq 1$, i.e., the quadrupole eigenvalue already is split into two quadratics in the first approximation.

C. Let the initial imperfection have the form

$$w^0(x,y)=\sum_{i,j=1}^{\infty}w_{ij}^0\sin iy\sin\frac{j\pi R}{L}x$$

and in addition $\lambda_{nm} \neq \lambda_{n+1,m}$, but let the two eigenfunctions

$$\Phi_{nm}^1=\gamma_{nm}\sin ny\sin\frac{m\pi R}{L}x,$$
$$\Phi_{nm}^2=\gamma_{nm}\cos ny\sin\frac{m\pi R}{L}x,\quad n=2,3,4,\ldots,m=1,2,3,\ldots$$

correspond to the last eigenvalue λ_{nm}. Let us introduce the following notation

$$T_{ij}=\left(ta\Phi_{nm}^i,\Phi_{nm}^j\right),\quad S_{ij}=\left(sd\Phi_{nm}^i,\Phi_{nm}^j\right),\quad A_i=\left(ad\Phi_{nm}^i,\Phi_{nm}^i\right),$$
$$D_i=\left(d^2\Phi_{nm}^i,\Phi_{nm}^i\right),\text{ where } i,j=1,2;$$

The determinant corresponding to the system (3.2) has the form

$$\begin{vmatrix} T_{11}D_1A_1^{-1}+S_{11}+\lambda^{(1)} & T_{21}D_1A_1^{-1}+S_{21} \\ T_{12}D_2A_2^{-1}+S_{12} & T_{22}D_2A_2^{-1}+S_{22}+\lambda^{(1)} \end{vmatrix}=0. \tag{4.5}$$

Since the curvatures w^{o}_{xx}, w^{o}_{yy}, and w^{o}_{xy} are continuous functions, then their corresponding series can be integrated term by term in the given domain. Taking account of the equalities

$$\int_0^{2\pi}\sin iy\sin^2 ny\,dy=0,\quad \int_0^{2\pi}\cos iy\cos ny\sin ny\,dy=0,$$

$$\int_0^{2\pi}\sin iy\cos^2 ny\,dy=0,$$

we can obtain $T_{11} = T_{21} = S_{11} = S_{22} = 0$. To calculate T_{21} the following must be noted

$$\int_0^{2\pi}\sin iy\cos ny\sin ny\,dy=\begin{cases}0, & i\neq 2n,\\ \frac{\pi}{2}, & i=2n,\end{cases}$$

$$\int_0^{2\pi}\cos iy\sin^2 ny\,dy=\begin{cases}0, & i\neq 2n,\\ -\frac{\pi}{2}, & i=2n,\end{cases}$$

$$\int_0^{L/R}\sin\frac{j\pi R}{L}x\sin^2\frac{m\pi R}{L}x\,dx=\begin{cases}0, & j\ \text{even},\\ \frac{4m^2L}{\pi Rj\,(4m^2-j^2)}, & j\ \text{odd},\end{cases}$$

$$\int_0^{L/R}\cos\frac{j\pi R}{L}x\cos\frac{m\pi R}{L}x\sin\frac{m\pi R}{L}x\,dx=\begin{cases}0, & j\ \text{even},\\ \frac{2mL}{\pi R\,(4m^2-j^2)}, & j\ \text{odd},\end{cases}$$

As a result of the calculations we obtain

$$T_{21}=2\,(nm\pi)^2\frac{h}{L}\left[n^2+\left(\frac{m\pi R}{L}\right)^2\right]^2\gamma_{nm}^2\sum_{j=1}^{\infty}\frac{w_{2n,(2j-1)}}{2j-1}.$$

The remaining quantities are calculated analogously. Finally, the determinant (4.5) takes the form

$$\begin{vmatrix} \lambda^{(1)} & -4\frac{h}{R}\left(\frac{R}{L}\right)^3(m\pi)^4n^2\gamma_{nm}^2\sum_{j=1}^{\infty}\frac{w^0_{2n,(2j-1)}}{2j-1} \\ -4\frac{h}{R}\left(\frac{R}{L}\right)^3(m\pi)^4n^2\gamma_{nm}^2\sum_{j=1}^{\infty}\frac{w^0_{2n,(2j-1)}}{2j-1} & \lambda^{(1)} \end{vmatrix}=0.$$

Therefore, if the mode $\sin 2ny\sin j/\pi Rx/L$, where j is odd, is present among the modes giving the initial incorrection, then the duplex eigenvalue λ_{nm} is split into simple eigenvalues in a first approximation.

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APPLICATION OF PHOTOELASTIC COATINGS TO STRAIN INVESTIGATION IN POLYCRYSTAL MICRODOMAINS

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A thin layer (coating) of optically active material is deposited on the specimen surface in investigations by the method of photoelastic coatings. If the coating thickness is relatively small, processing the experiment results offers no special difficulties since the optical quantities obtained during the experiment will be proportional to the measured strains on the specimen surface [1]. If the coating thickness is commensurate with the characteristic dimension of the strain zone, then analysis of the measurement results is complicated substantially and requires special processing methods (refinements).

This problem is especially complex for strain investigations in microdomains of real polycrystalline materials. Here even for small loads (in the domain of the so-called microplastic strains) inelastic deformation occurs by the formation and development of displacements governed by the localization of slip traces. Such zones of local strain concentration have, in turn, a finer structure and can reflect the result of the action of several strain mechanisms in the slip band domain [2, 3]. In all these cases the minimal coating thickness realizable in practice exceeds the size of the section deformed and the measurement results cannot therefore be used without appropriate correction.

Different cases of strain measurement in a slip band domain of width 2a (Fig. 1) are considered in this paper. The thickness d of the photoelastic coating being used (not shown in Fig. 1) considerably exceeds the deformation zone dimension ($d > 2a$). We hence consider the strains homogeneous in the slip band domain while they can be neglected outside this zone. We denote the projections of the displacement vector P_0 characterizing the displacement of the undeformed sections as a rigid whole on the x, y, z axes by U_0, V_0, W_0. We consider the displacements U, V, W within the deformed zone $-a \leqslant x \leqslant a$ linear functions of the coordinate x.

1. Out of all the displacement vector component, let just the vertical component V_0 be different from zero. If the length of the slip band is large in the z axis direction, it can be considered that the coating deposited on the surface y = 0 (see Fig. 1) is under plane strain conditions with the following boundary conditions:

$$\text{for } y = d \ \sigma_x = 0,\ \tau_{xy} = 0, \qquad (1.1)$$
$$\text{for } y = 0 \quad V = \begin{cases} -V_0, & \text{if } -\infty < x < -a, \\ V_0 x/a, & \text{if } -a \leqslant x \leqslant a, \\ V_0, & \text{if } a < x < \infty \end{cases}$$

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